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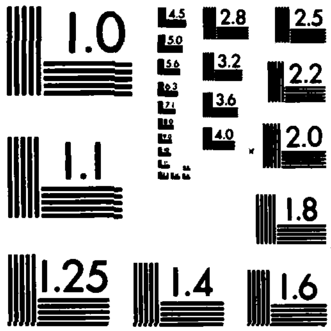

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PORTLAND STATE UNIVERSITY
p.o. box 751
portland, oregon 97207
PSU
(503) 229-4281
(503) 229-4290

DR. HERMAN J. MIGLIORE, P.E.
Professor of Mechanical Engineering

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A global collocation procedure was attempted on two-D Navier-Stokes equations, but suffered from significant ringing. Historically, multi-element collocation has been based on Hermit shape functions for Navier-Stokes. Alternative, a weak formulation Navier-Stokes solution was developed using biquadratic Lagrangian functions on element boundaries and discrete Galerkin (collocation) expressions on interiors. To account for pressure, a penalty function expression was evaluated as part of a weighted integral, using bilinear shape functions.

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21. This method was exercised on flow through a duct that included a pivoting flap that protruded into the flow where the flap motion and fluid flow interacted.

C^0 Collocation Solution For Weak-Formulation Navier-Stokes Equations

H.J. Migliore, E.T. Ray, E.G. McReynolds

Portland State University
Portland, OR 97207

INTRODUCTION

Collocation was used to solve a nonlinear wave equation as applied to the analysis of cable dynamics including hydrodynamic loading, surface excitation, and change of length with time (1). The collocation formulation was global in nature and was cast in terms of expressions for spatial derivatives which were directly substituted in the differential equations of motion. This approach suggested a generality for application of collocation which was not particularly sensitive to the order of the differential equation, to nonlinearities, or to complex boundary equations, since the latter were expressed explicitly along with the system of residual equations at interior collocation points.

As an extension to this work, the modelling of real fluids, i.e. solution two-dimensional Navier-Stokes equations, were of interest especially in terms of interactive boundary conditions, such as encountered in structure-fluid interaction. In light of addressing interaction, a primative variable version of Navier-Stokes equation was adopted apriori, rather than stream or vorticity stream function approaches which might interfere with the flexibility required along boundaries (2). For the sake of economy in two-D and simplicity (3,4), a penalty function formulation was employed. The solution of these two-dimensional Navier-Stokes equations was to be based on collocation, using the previous global procedure as a starting-off point. This paper discusses a collocation solution for the Navier-Stokes equations including the historical evolution that led to the developed procedure: discrete-Galerkin collocation on the interior and weak-formulation Galerkin using FEM shape functions on element boundaries.

GLOBAL COLLOCATION

Several reviews and applications of collocation can be found in Villadsen and Michelsen (5), Finlayson (6,7), Pinder (8,9,10), and Becker, Carey and Oden (11,12), and only an overview will be given here. Collocation's theoretical base is Method of Weighted Residual (MWR) where a trial solution is substituted into the differential equation, resulting in a residual expression reminiscent of the given differential equation. The residual is minimized by weighting over the domain. In the case of Galerkin, the weighting functions are the basis functions associated with the trial solution, and in the case of collocation, the Dirac delta has historically been interpreted as the weighting function.

The optimal collocation points are not generally arbitrary or equally spaced. One approach (5,6,12) uses orthogonal trial functions, such as Legendre, in the Galerkin integral. Gaussian quadrature (12) is then employed with two consequences: collocation is linked to discrete Galerkin, and collocation points are determined. By thus minimizing the residual, expressions are hypothetically available for determining coefficients of the trial solution, analogous to a Rayleigh-Ritz or Least-Squares approach. The flexibility of collocation, however, is to not directly consider $R(x_i)=0$, but rather to develop expressions for derivatives based on using orthogonal polynomials in a discrete Galerkin approach.

Starting with a general trial solution with three terms,

$$u(x) = \sum_{k=1}^{3} a_k P_k(x) \qquad [1]$$

$$u(x)P_i(x)dx = \sum_{k=1}^{3} a_k P_k(x) P_i(x)$$

$$\int_{-1}^{1} u(x)P_i(x)dx = \int_{-1}^{1} \sum_{k=1}^{3} a_k P_k(x) P_i(x) dx$$

Employing Lobatto quadration,

$$\sum_{j=1}^{3} w_j u(x_j) P_i(x_j) = \sum_{j=1}^{3} w_j \sum_{k=1}^{3} a_k P_k(x_j) P_i(x_j)$$

$$= \sum_{k=1}^{3} a_k \sum_{j=1}^{3} w_j P_k(x_j) P_i(x_j) = a_i \sum_{j=1}^{3} w_j P_i^2(x_j)$$

Therefore:

$$a_k = \frac{\sum_{j=1}^{3} w_j u(x_j) P_k(x_j)}{\sum_{j=1}^{3} w_j P_k^2(x_j)}$$

Noting that $x_1 = -1,\ x_2 = 0,\ x_3 = 1;\quad w_1 = \frac{1}{3},\ w_2 = \frac{4}{3},\ w_3 = \frac{1}{3}$

Taking the derivative of the trial solution

$$\frac{\partial u(x)}{\partial x} = \sum_{k=1}^{3} a_k \frac{\partial P_k}{\partial x}$$

Knowing P_k and its derivative as a function of x and knowing a_k as a function of $u(x_j)$, a derivative expression can be found as a function of $u(x_j)$ and x:

$$\frac{\partial u}{\partial x} = [A]u \quad \text{where an element in matrix A is } A_j = A\{a_k(u_j)\} \qquad [2]$$

For a derivative at a specific point, i:

$$\frac{\partial u(x_i)}{\partial x} = A_{ij}\ u(x_j)$$

Similarly,

$$\frac{\partial u^2}{\partial x^2} = [A]\frac{\partial u}{\partial x} = [A]^2 u \qquad [3]$$

Rather than dealing with integrals or residuals directly, Equations 2 and 3 are substituted into the PDE, resulting in a system of ODE's. Equations 2 and 3 are used in either the x or y direction for the two dimensional case. For mixed derivatives in two-D, Equation 2 is used twice, relative to x and y. In applying these expressions boundary conditions must be incorporated in the form of Neumann or Dirichlet for second order problems. For a specified derivative, the inverse of Equation 2 was used to determine the value of $u(x_k)$, x_k at the boundary, and, as in the Dirichlet case, the specified value of $u(x_k)$ could then be used in Equations 2 and 3 for determining $u(x_i)$ at other Lobatto points.

Global collocation, using derivative expressions similar to Equations 2 and 3, was attempted on penalty function Navier-Stokes. The test problem was a driven cavity with conventional boundary conditions, i.e. no-slip on three walls. In general, global collocation suffered from significant ringing whenever sharp gradients were encountered and required small time integration steps when collocation points were closely packed. A multi-element approach was needed in order that one trial function was not forced to satisfy sharp gradients of the field variable, at boundaries in the case of the driven cavity.

MULTI-ELEMENT COLLOCATION

Historically, elementary approaches in collocation were a direct extension of global techniques, in that collocation points were on interiors of elements, and interelement points provide local boundary conditions. In effect, additional expressions were needed to determine field variable values at interelement points and could be derived by considering C^1 continuity (5,6) between elements for second order problems. A slightly different approach was successfully attempted by the authors on one-dimensional advection-diffusion where boundary expressions were developed from C^0 and C^1 considerations (13). For two-D Navier-Stokes equations, a conventional C^1 collocation was attempted, and interelement expressions were found to be cumbersome, thus interfering with making changes in defining mesh geometry and complicating the prospects of incorporating distorting elements.

Alternatively, finite element functions have been used as trial functions in conjunction with collocation. For second order problems, C^1 is required between elements (12), and Hermite shape functions (8,12) were used to automatically satisfy this requirement. Lagrangian shape functions (7,12) have also been considered, but require an additional set of expressions related to the C^1 requirement.

In either case, the C^1 requirement must be definitely enforced, and the complexity of collocation finite element equations are increased (12) relative to Galerkin finite element (GFEM). GFEM can be integrated by parts resulting in less stringent C^0 between elements. However, the Dirac delta weighting function associated with collocation cannot be differentiated preventing a weak MWR formulation. Because distorting elements would be required for future interaction work, simple element formulation was necessary, and a C^0 approach was preferred.

C^0 FEM COLLOCATION FOR NAVIER-STOKES

Galerkin condition applied to Navier-Stokes equations:

$$\int_\Omega \left[\frac{D\underset{\sim}{u}}{Dt} - \nabla\cdot\underset{\sim}{T} \right] N d\Omega = 0 \qquad [4]$$

and N are shape functions of the form $N_i=1$ at $(x,y)=(x_i,y_i)$ and $N_i=0$ at $(x,y)=(x_j,y_j)\neq(x_i,y_i)$. Using $\nabla\cdot(N\underset{\sim}{T}) = N\nabla\cdot\underset{\sim}{T} + \nabla N\cdot\underset{\sim}{T}$,

$$\int_\Omega \frac{D\underset{\sim}{u}}{Dt} N\, d\Omega + \int_\Omega \left[\nabla N\cdot\underset{\sim}{T} - \nabla\cdot(N\underset{\sim}{T})\right] d\Omega = 0$$

and using the Divergence Theorem, the weak form results:

$$\int_\Omega \left[\frac{D\underset{\sim}{u}}{Dt} N + \underset{\sim}{T}\cdot\nabla N\right] d\Omega - \int_{\partial\Omega} N\, \underset{\sim}{T}\cdot\underset{\sim}{n}\, ds = 0 \qquad [5]$$

The second integral, the "natural" boundary condition, represents flux across $\partial\Omega$ (12). If N are Lagrangian finite element functions, C^0 will be required between elements and the Galerkin condition becomes (12):

$$\sum_e \int_{\Omega^e} \left[\frac{D\underset{\sim}{u}}{Dt} N + \underset{\sim}{T}\cdot\nabla N\right] d\Omega^e - \int_{\partial\Omega} N\, \underset{\sim}{T}\cdot\underset{\sim}{n}\, ds = 0 \qquad [6]$$

where e are finite elements within $\partial\Omega$. Equation 6 is the basis for a classical Galerkin-FEM approach, and is employed on element boundaries by considering N_i for (x_i,y_i) only on element boundaries. Alternatively, if we selectively reverse the integration-by-parts:

$$\sum_e \int_{\Omega^e} \left[\frac{D\underset{\sim}{u}}{Dt} N + \nabla\cdot(N\underset{\sim}{T}) - \nabla\cdot\underset{\sim}{T}\, N \right] d\Omega^e - \int_{\partial\Omega} N\, \underset{\sim}{T}\cdot\underset{\sim}{n}\, ds = 0$$

$$\sum_e \int_{\Omega^e} \left[\frac{D\underset{\sim}{u}}{Dt} - \nabla\cdot\underset{\sim}{T}\right] N d\Omega^e + \sum_e \int_{\partial\Omega^e} N\, \underset{\sim}{T}\cdot\underset{\sim}{n} ds - \int_{\partial\Omega} N\, \underset{\sim}{T}\cdot\underset{\sim}{n} ds = 0 \qquad [7]$$

Equation 7 is the basis for our C^0 collocation approach and is used on the interior of elements by considering N_i for (x_i,y_i) only on interiors, resulting in:

$$\sum_e \int_{\Omega^e} \left[\frac{D\underset{\sim}{u}}{Dt} - \nabla\cdot\underset{\sim}{T}\right] N d\Omega^e = 0$$

Lobatto quadrature results in the discrete Galerkin expression:

$$\sum_e \{\sum_i w_i [\frac{D\underset{\sim}{u}}{Dt} - \nabla\cdot\underset{\sim}{T}] N\} = 0 \qquad [8]$$

$$R = \frac{D\underset{\sim}{u}}{Dt} - \nabla\cdot\underset{\sim}{T} = 0 \quad \text{evaluated on interior collocation points} \qquad [9]$$

This technique was implemented on Navier-Stokes equations and will be detailed in following sections. The specific form of Navier-Stokes equations that was solved will first be reviewed.

PENALTY FUNCTION NAVIER STOKES

As mentioned, the problem of interest was the solution to the two-D Navier-Stokes equations. Because of compactness (3) and economy in two-D (4), a penalty function approach was adopted. The formulation will be outlined below; further details can be found in Hughes (3) and Oden (14).

Navier-Stokes equations can be written as (15,16):

$$\rho \frac{D\underset{\sim}{u}}{Dt} = \nabla\cdot(\underset{\sim}{T})$$

for incompressible fluid, where

$$\underset{\sim}{T} = \begin{bmatrix} -p + 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & -p + 2\mu\frac{\partial v}{\partial y} \end{bmatrix}$$

For the incompressible case, continuity requires the additional expression $\nabla\cdot u=0$, but for the penalty function approach, continuity is approximately satisfied by:

$$p = -\lambda \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right]$$

Noting that an expression for p is available, the system to be solved becomes:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-p + 2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad [10]$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(-p + 2\mu\frac{\partial v}{\partial y}\right) \qquad [11]$$

$$p = -\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad [12]$$

COLLOCATION SOLUTION OF NAVIER-STOKES EQUATIONS

As discussed, the spatial derivative expressions developed from our collocation approach can be directly substituted in Equations 10, 11, and 12. The question of boundary conditions, whether local in the sense of interelement coupling, or global in the sense of inflow/outflow requirements for a physical problem, could not be effectively answered by utilizing C^1 continuity. After considerable numerical experimentation, the collocation approach that yielded reasonable geometry flexibility and convergence properties, was collocation on the interior and a finite element approach on the boundary of elements and of the domain.

A nine node element was used (15,11). The central node, number 9, was the only collocation point. Derivative expressions, Equations 2 and 3 were applied to Equations 10 to 12. Matrix [A] has been subscripted and has been given the designation [B] when differentiation in the y direction was desired. Repeated subscript indicates summation; for Equation 10:

$$\dot{u}_9 + u_9 A_{9k} u_k + v_9 B_{9k} u_k = -\frac{1}{\rho} A_{9m} p_m + 2\frac{\mu}{\rho} A_{9m} A_{mk} u_k + \frac{\mu}{\rho} B_{9m} B_{mk} u_k + \frac{\mu}{\rho} B_{9m} A_{mk} v_k$$

and $$\frac{-p_m}{\rho} = \frac{\lambda}{\rho} D_{m\ell} A_{\ell k} u_k + \frac{\lambda}{\rho} D_{m\ell} B_{\ell k} v_k$$

$m = 1,2, \ldots 9 \qquad \ell = 1,2, \ldots 9 \qquad k = 1,2, \ldots 9$

The derivation of the [D] matrix is detailed in Appendix I. Substituting p_m, gives:

$$\dot{u}_9 + u_9 A_{9k} u_k + v_9 B_{9k} u_k = \Big[\frac{\lambda}{\rho} A_{9m} D_{m\ell} A_{\ell k} + \frac{2\mu}{\rho} A_{9m} A_{mk} + \frac{\mu}{\rho} B_{9m} B_{mk}\Big] u_k + \Big[\frac{\lambda}{\rho} A_{9m} D_{m\ell} B_{\ell k} + \frac{\mu}{\rho} B_{9m} A_{mk}\Big] v_k \qquad [13]$$

This time varying equation became part of a larger system that included boundary equations relative to elements and domain boundaries. The discussion on development of these boundary equations and the time solution are given in the next sections.

FEM COUPLING EQUATIONS

The Galerkin FEM expressions for penalty function Navier-Stokes equations are:

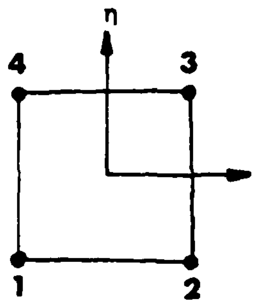
$$\int_\Omega \Big\{\rho\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\Big) - \frac{\partial}{\partial x}\Big(-p + 2\mu\frac{\partial u}{\partial x}\Big) - \frac{\partial}{\partial y}\mu\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big)\Big\} \Psi_i d\Omega = 0 \qquad [14]$$

$$\int_\Omega \Big\{\rho\Big(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\Big) - \frac{\partial}{\partial x}\mu\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(-p + 2\mu\frac{\partial v}{\partial y}\Big)\Big\} \Psi_i d\Omega = 0 \qquad [15]$$

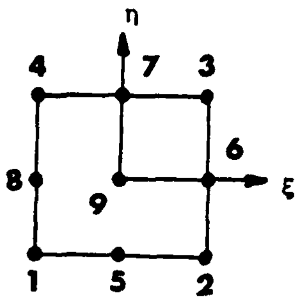
$$\int_{\Omega^e} \Big\{p + \lambda\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big)\Big\} \phi_i^e d\Omega^e = 0 \qquad [16]$$

Ω denoting global minimization and Ω^e denoting local minimization (implying C^{-1} between elements). Figure 1 defines Ψ and ϕ shape functions over an element (11).

The utilization of Equation 16 is described in Appendix I. Equation 14 is



$$\phi_1 = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$\phi_2 = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$\phi_3 = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$\phi_4 = \tfrac{1}{4}(1-\xi)(1+\eta)$$



$$\psi_1 = \tfrac{1}{4}(\xi^2-\xi)(\eta^2-\eta)$$
$$\psi_2 = \tfrac{1}{4}(\xi^2+\xi)(\eta^2-\eta)$$
$$\psi_3 = \tfrac{1}{4}(\xi^2+\xi)(\eta^2+\eta)$$
$$\psi_4 = \tfrac{1}{4}(\xi^2-\xi)(\eta^2+\eta)$$
$$\psi_5 = \tfrac{1}{2}(1-\xi^2)(\eta^2-\eta)$$
$$\psi_6 = \tfrac{1}{2}(\xi^2+\xi)(1-\eta^2)$$
$$\psi_7 = \tfrac{1}{2}(1-\xi^2)(\eta^2+\eta)$$
$$\psi_8 = \tfrac{1}{2}(\xi^2-\xi)(1-\eta^2)$$
$$\psi_9 = (1-\xi^2)(1-\eta^2)$$

a) Bilinear used for pressure b) Biquadratic used for velocity

FIGURE 1. Shape functions

integrated by parts so as to develop a weak formulation version, as was done in Equation 6:

$$\int_{\Omega^e}\left[\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\Psi_i + \frac{1}{\rho}\left(-p + 2\mu\frac{\partial u}{\partial x}\right)\frac{\partial \Psi_i}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\frac{\partial \Psi_i}{\partial y}\right]d\Omega^e = \frac{1}{\rho}\,\beta_u^i$$

The boundary integral, β_u^i, will be further discussed in Appendix II. Using Lobatto quadrature, not summing on i, gives:

$$W_i(\dot{u}_i + u_i A_{ik} u_k + v_i B_{ik} u_k)$$

$$+ W_\alpha\left\lfloor\frac{-p_\alpha}{\rho}\, a_\alpha^i + \frac{2\mu}{\rho}\, a_\alpha^i A_{\alpha k} u_k + \frac{\mu}{\rho}\, b_\alpha^i B_{\alpha k} u_k + \frac{\mu}{\rho}\, b_\alpha^i A_{\alpha k} v_k\right\rfloor = \frac{1}{\rho}\,\beta_u^i$$

From Appendix I, $D_{\alpha j} = \phi_{\alpha\beta} m_{\beta i} W_j \phi_{ji}$, not summing on i, then,

$$W_i(\dot{u}_i + u_i A_{ik} u_k + v_i B_{ik} u_k) = -\left\lfloor\frac{\lambda}{\rho} W_\alpha a_\alpha^i D_{\alpha j} A_{jk} + \frac{2\mu}{\rho} W_\alpha a_\alpha^i A_{\alpha k} + \frac{\mu}{\rho} W_\alpha b_\alpha^i B_{\alpha k}\right\rfloor u_k$$

$$-\left\lfloor\frac{\lambda}{\rho} W_\alpha a_\alpha^i D_{\alpha j} B_{jk} + \frac{\mu}{\rho} W_\alpha b_\alpha^i A_{\alpha k}\right\rfloor v_k + \frac{1}{\rho}\,\beta_u^i \qquad [17]$$

where $i = 1,..8$; $\alpha,j,k = 1,..9$; W_i are 2-D weights defined in Appendix III. Similarly, the v velocity version of Equation 17 was developed and these equations, along with Equation 13, and its v velocity version were then solved in time.

IMPLEMENTATION

Assembly

Equations 13, 17 and their "v" counterparts were applied to a master element. The domain was composed of master elements which were patched together. For example, the equation governing a midside node, node 6 on element e coincident with node 8 on element e+1 would be:

$$\int_{\Omega^e} \left[\rho \frac{D\underline{u}}{Dt} \Psi_6 + \nabla\Psi_6 \cdot (\underline{T})\right] dxdy + \int_{\Omega^{e+1}} \left[\rho \frac{D\underline{u}}{Dt} \Psi_8 + \nabla\Psi_8 \cdot (\underline{T})\right] dxdy = 0 \qquad [18]$$

The boundary integral β would only apply on the domain boundary for nodes that were part of the system degrees of freedom, e.g.in-flow, outflow boundaries. No slip boundaries were treated as constrained nodes with specified velocities and were not part of the degrees of freedom. Looking back at the integration by parts, Equation 7, the flux between elements does not appear as additional terms to Equation 18 because the flux is generally equal and opposite from each element. Only in the case of an applied flux at an element boundary, $\int_{\partial\Omega^e} N\, \underline{T}\cdot\underline{n}\, ds$, would Equation 17 need to be modified.

Time Varying Equations

Because of the stiff terms associated with the penalty parameter, an appropriate ODE solver was required. LSODE (17) was employed, a subroutine based on Gear's algorithm, with an Euler start-up procedure.

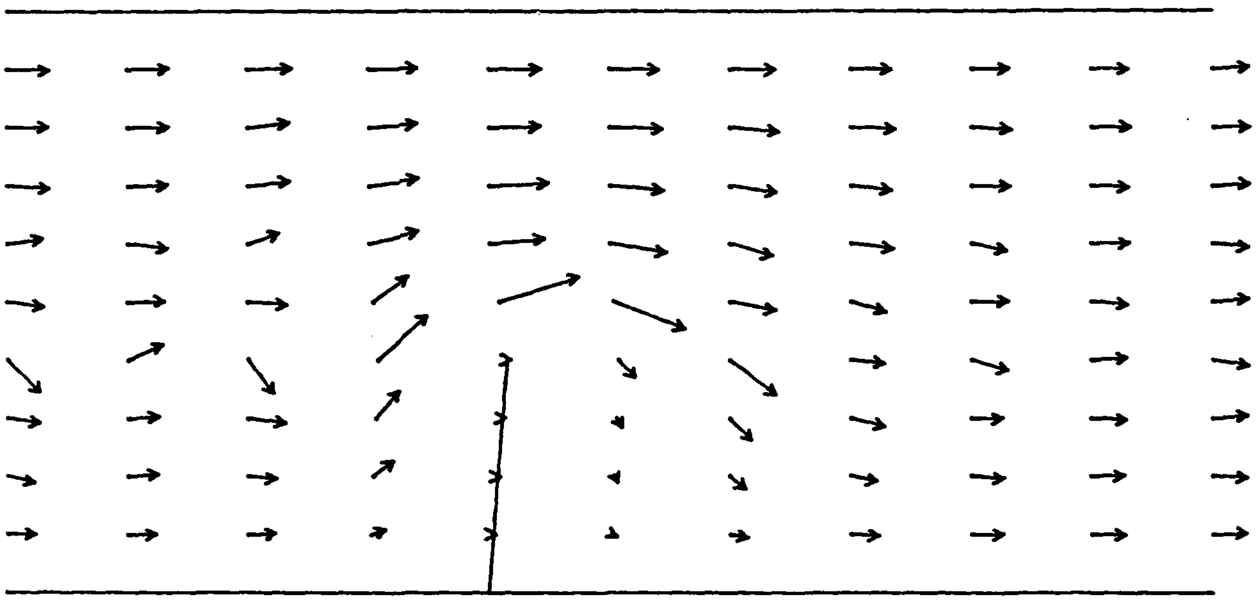
Applications

The C^0-Collocation program was exercised on several time varying flows (18). Results consist of velocity vector fields at t = 1.5 seconds, for a moving flap responding to fluid induced forces and spring restoration forces. Kinematic viscosity in all cases was approximately that of water: $\upsilon = 1.3 \times 10^{-6}\ m^2/sec$. Figures 2 and 3 show a duct which is 0.15 m high by 0.30 m long composed of nine node elements 0.03 m high and 0.06 m long. Figure 4 shows a duct which is 0.15 m high by 3.5 m long composed of nine node elements 0.03 m high and length from left to right on the figure of 0.95, .05, .05, .05, and 2.4 m respectively. Each duct had a 0.06 meter high pivoting, rigid flap with a specific density of 5.0 and thickness represented by a line. A torsional spring is located at the base of the flap, with spring stiffness 0.0023 N/rad. Boundary conditions for the velocity results shown in Figures 2-4 are:

Top, bottom, sides of flap: $u = 0, \quad v = 0$

		Figures 2, 4	Figure 3
Inlet:	$\frac{\beta_u^i}{W_i \rho} = 0.31 \cos(\pi t)\ m/sec^2$;	$\beta_v^i = 0$	$v = 0$
Outlet:	$\beta_u^i = 0$;	$\beta_v^i = 0$	$\beta_v^i = 0$

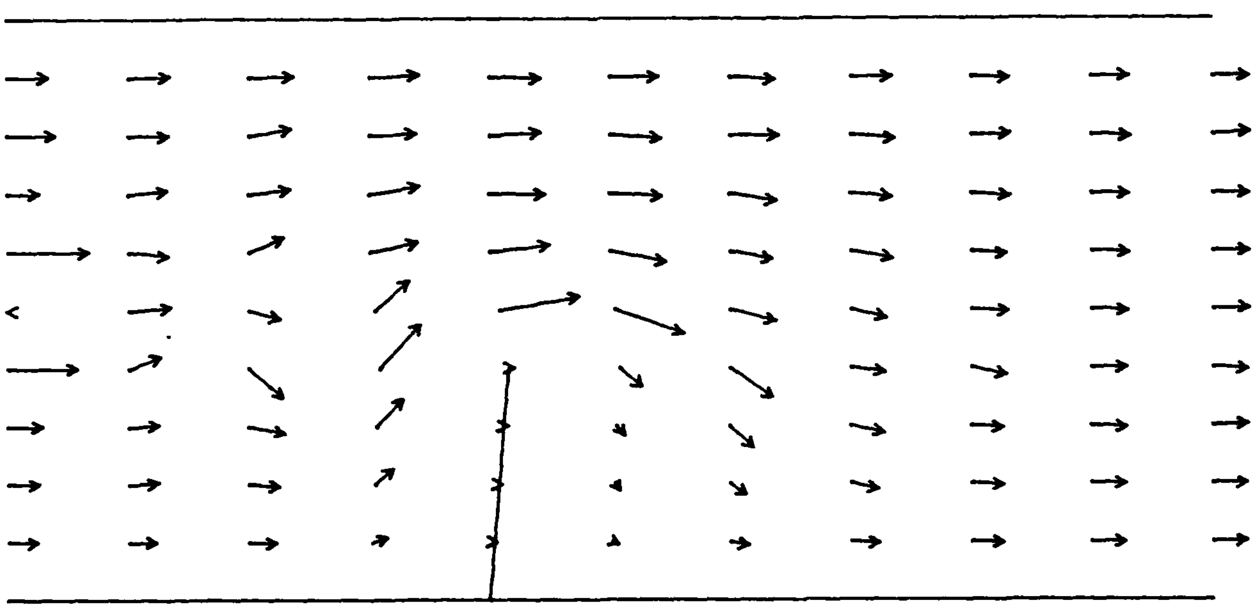
The equations governing flap motion included fluid loading and in turn provided changing geometry for the fluid boundary. The flap equations along with the Navier-Stokes equations were solved simultaneously. In order to more closely model a structure-fluid interaction problem, elements were allowed to distort as a function of flap rotation. Flap geometry was approximated by using bilinear shape functions, making them subparametric relative to the velocity functions. This distortion required modification to the derivative expressions as was described for the One-D case. The transformation from One-D weights, w_i, to Two-D weights, W_i, included the effects of distortion.



$\theta = 0.037$ $\dot{\theta} = 0.013$ 0.10 m/sec =

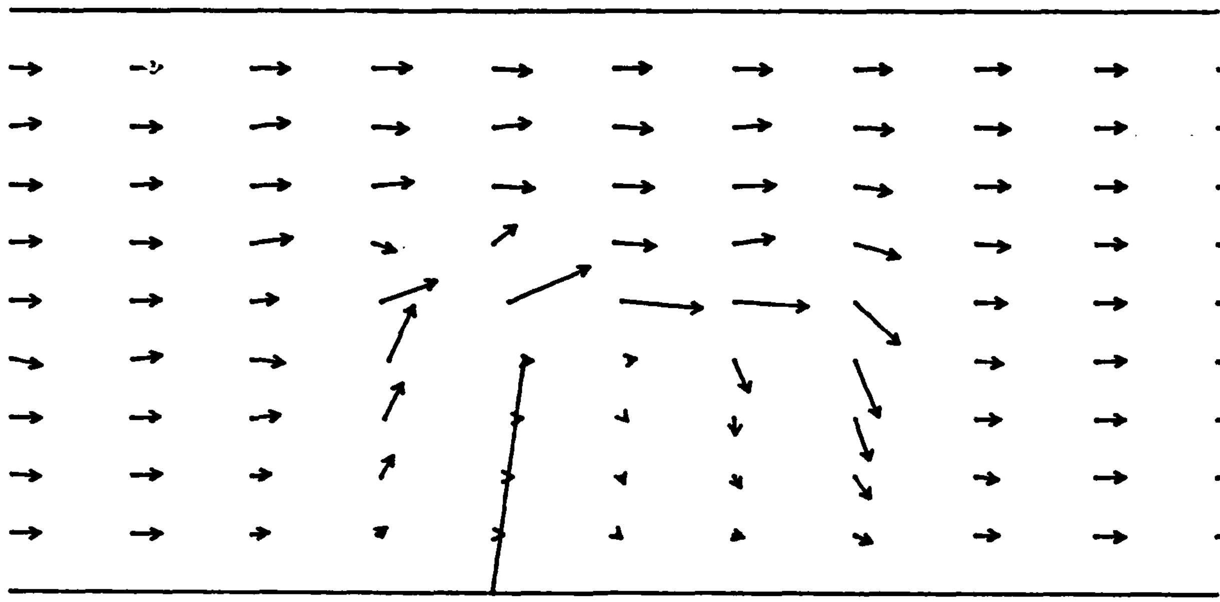
FIGURE 2. Interactive flap velocity field, inlet $\beta_v = 0$

Figure 4 incorporated longer elements on the inlet and outlet than on the interior, and these variations seem to significantly affect stability and the response of the flap. Both Figures 2 and 3 show instabilities above and upstream from the top of flap, and these instabilities extend to the inlet. In the case of specifying tangential surface traction, β_v, instabilities are manifested in inlet v-component velocity. In the case of specifying v velocity, instabilities at the inlet are manifested in u-component velocity. Figure 4 shows less instability due to relocating the inlet farther from the flap.



$\theta = 0.04$ $\dot{\theta} = 0.019$ 0.10 m/sec =

FIGURE 3. Interactive flap velocity field, inlet v = 0



θ = 0.063 $\dot{\theta}$ = 0.034   0.10 m/sec = ⟶

FIGURE 4. Interactive flap velocity field, inlet β_V = 0 and long elements

NODALLY INTEGRATED GFEM

No additional attempts were made to control instabilities such as incorporating upwinding procedures (19,20), or refining the mesh (15), as has been done with Galerkin finite element. These modifications were not made because this particular C^0-collocation approach performs the same as nodally integrated GFEM and as a consequence will suffer from similar instability problems and will respond to similar fixes. Regarding similarities, Equation 17 and its "v" counterpart were employed for all nine nodes of the master element with virtually identical results as when using Equation 13 for node 9 and Equation 17 for nodes 1 through 8.

CONCLUSIONS

Multi-element collocation was successfully applied to the solution of the two-dimensional, unsteady Navier-Stokes equations. In order to take advantage of the simplicity of C^0 between elements, a weak formulation version of the Navier-Stokes equations was desired, which precluded viewing collocation as based on a Dirac delta weighting function. The developed C^0 collocation procedure evaluated residuals on element interiors based on using a quadrated version of the MWR integral. Element boundaries and coupling were treated as in Galerkin FEM. Effectively, the developed C^0 collocation technique is equivalent to GFEM (21). Because of the normal (explicit) form of the derivative expressions and their evaluation at Lobatto points, the similarity between techniques is limited to GFEM that incorporates low order shape functions, mass lumping, and nodal integration (19) rather than Gaussian integration. Because of this equivalency, no additional comparisons were made between collocation and GFEM relative to finite difference or similar techniques (3,4,15,20). The advantages of this C^0 collocation technique are:

1. nodal values are calculated without using additional expressions for quadrature and interpolation

2. the system of time equations are in normal (explicit) form simplifying the incorporation of other governing equations as encountered in an interaction problem
3. shape functions are not differentiated; rather derivative expressions based on Lobatto quadrature and Legendre polynomials were developed facilitating the mixing of shape functions as is required with penalty function approach, and facilitating the quadration of integral equations: both interior GFEM Navier-Stokes equations and boundary integrals
4. derivative expressions are used directly in residual equations, specified derivatives and other derivative equations.

These simplifying characteristics proved to be significant in attempting to model and solve fluid interaction.

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APPENDIX I. Theory for "D" Matrix

Galerkin integral for penalty expression (12,14):

$$\int_{\Omega^e} \{p + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\} \phi_i d\Omega^e = 0 \qquad i = 1,\ldots 4, \text{ bilinear shape functions}$$

Using Lobatto quadrature gives:

$$W_j p_j \phi_{ji} + \lambda W_j (A_{jk} u_k + B_{jk} v_k) \phi_{ji} = 0 \qquad j = 1,\ldots 9, \text{ Lobatto points}$$

where Ω_e domain is restricted t subdomain element, e.

$$p_j = \sum_{k=1}^{4} \phi_{jk} p_k \qquad j = 1,\ldots 9$$

$$W_j \phi_{jk} p_k \phi_{ji} = -\lambda W_j (A_{jk} u_k + B_{jk} v_k) \phi_{ji}$$

$$M^{ik} p_k = -\lambda W_j \phi_{ji} A_{jk} u_k - \lambda W_j \phi_{ji} B_{jk} v_k \qquad \text{where } M^{ik} = W_j \phi_{ij} \phi_{jk}$$

$$[\phi_{jk}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \qquad \text{i.e. } \phi_k \text{ evaluated at point } j$$

Using repeated subscripts to indicate summation:

$$m_{\alpha i}M^{ik}p_k = -\lambda m_{\beta i}W_j\phi_{ji}A_{jk}u_k - \lambda m_{\beta i}W_j\phi_{ji}B_{jk}v_k$$

Where $m_{\beta i}M^{ik} = \delta^k_\beta$, and δ^k_β is Kronecker delta,

$$p_\alpha = \phi_{\alpha\beta}p_\beta = -\lambda\phi_{\alpha\beta}m_{\beta i}W_j\phi_{ji}A_{jk}u_k - \lambda\phi_{\alpha\beta}m_{\beta i}W_j\phi_{ji}B_{jk}v_k$$

$$D_{\alpha j} = \phi_{\alpha\beta}m_{\beta i}W_j\phi_{ji}$$

APPENDIX II. Boundary Conditions

Domain boundary conditions can be a combination of given velocities or specified surface traction force. Velocities are defined on that portion of the boundary $\partial\Omega_u$, and surface tractions are defined on $\partial\Omega_h$, such that:

$$\partial\Omega = \partial\Omega_h \cup \partial\Omega_u \qquad \partial\Omega_h \cap \partial\Omega_u = \emptyset$$

Looking at the boundary integral, β; $\underset{\sim}{n}$ is outward normal to $\partial\Omega_h$:

$$\beta^i_u = \int_{\partial\Omega_h} N_i\{T_{11}\ \underset{\sim}{i}\cdot\underset{\sim}{n} + T_{12}\ \underset{\sim}{j}\cdot\underset{\sim}{n}\}\ ds$$

for $\underset{\sim}{n}$ oriented along the x-axis,

$$\beta^i_u = \int_{\partial\Omega_h} N_i\ T_{11}\ ds$$

specifying the normal traction component for the inflow boundary, $T_{11}=k$:

$$\beta^i_u = \int_{\partial\Omega_h} N_i\ k\ ds = w_i\ k$$

For the outflow boundary, the normal component will be specified as zero, $T_{11} = 0$:

$$\beta^i_u = 0$$

Considering the boundary integral for the second Navier-Stokes equation:

$$\beta^i_v = \int_{\partial\Omega_h} N_i\{T_{21}\underset{\sim}{i} + T_{22}\underset{\sim}{j}\}\cdot\underset{\sim}{n}ds$$

Again with $\underset{\sim}{n}$ oriented along x axis,

$$\beta^i_v = \int_{\partial\Omega_h} N_i\ T_{21}\ ds = w_i T_{21}$$

when the shear traction component is specified as zero, $T_{21} = 0$:

$$\beta^i_v = 0$$

APPENDIX III. Integration Weights

The master element is defined in terms of coordinates, ξ and η, on a global Cartesian coordinate framework, x,y. The integration weights must also be modified as follows:

$$\int_{\Omega_e} f(x,y)dxdy = {}_{-1}\int^{1}{}_{-1}\int^{1} F(\xi,\eta)\left|J\right|d\xi d\eta$$

$$= \left|J\right| {}_{-1}\int^{1} \sum_{i=1}^{3} w_i F(\xi_i,\eta)d\eta \quad = \left|J\right| \sum_{i=1}^{3} \sum_{j=1}^{3} w_i w_j F(\xi_i,\eta_j)$$

defining:

$$W_1 = \left|J\right| w_1 w_1 \qquad W_4 = \left|J\right| w_1 w_3 \qquad W_7 = \left|J\right| w_2 w_3$$
$$W_2 = \left|J\right| w_3 w_1 \qquad W_5 = \left|J\right| w_2 w_1 \qquad W_8 = \left|J\right| w_1 w_2$$
$$W_3 = \left|J\right| w_3 w_3 \qquad W_6 = \left|J\right| w_3 w_2 \qquad W_9 = \left|J\right| w_2 w_2$$

where $\left|J\right| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$

where w_1, w_2, w_3 are one-dimensional Lobatto weights. Thus, integration of any function f(x,y) e.g. u,v, has an associated Lobatto quadrature expression in terms of modified weights $W_1 \ldots W_9$ and functions $F(\xi,\eta)$ on the master element.

REFERENCES

1. Migliore, H. and McReynolds, E., "Ocean Cable Dynamics Using an Orthogonal Collocation Solution," AIAA Journal, Vol. 20, No. 8, pp. 1084-1091, 1983.

2. Katona, M. and Migliore, H., "Solid-Fluid Interaction with Finite Element Method," Naval Civil Engineering Lab, Pt. Heuneme, CA, Technical Memorandum, No. M-44-77-6, 1977.

3. Hughes, T.J.R., Liu, W. and Brooks, A., "Review of Finite Element Analysis of Incompressible Viscous Flows by the Penalty Function Formulation," Journal of Computational Physics, Vol. 30, No. 1, pp. 1-60, 1979.

4. Sani, R., Eaton, B., Gresho, P., Lee, R. and Chan, S., "On the Solution of the Time-Dependent Incompressible Navier-Stokes Equations Via a Penalty Galerkin Finite Element Method," Lawrence Livermore Lab, Livermore, CA, UCRL-85354, 1981.

5. Villadsen, J. and Michelsen, M., Solution of Differential Equation Models by Polynomial Approximation, Prentice-Hall, Englewood Cliffs, NJ, 1978.

6. Finlayson, B., The Method of Weighted Residuals and Variational Principles, Academic Press, New York, NY, 1972.

7. Finlayson, B., Nonlinear Analysis in Chemical Engineering, McGraw-Hill, New York, NY, 1980.

8. Pinder, G., and Shapiro, A., "A New Collocation Method for the Solution of the Convection-Dominated Transport Equation," Water Resources Research, Vol. 15, No. 5, pp. 1177-1182, 1979.

9. Lapidus, L. and Pinder, G., Numerical Solution of Partial Differential Equations in Science and Engineering, Wiley-Interscience, New York, NY, 1982.

10. Botha, J., and Pinder, G., "Fundamental Concepts in the Numerical Solution of Differential Equations," Wiley-Interscience, New York, NY, 1983.

11. Becker, E., Carey, G. and Oden, J.T., Finite Elements, An Introduction, Volume I, Prentice-Hall, Englewood Cliffs, NJ, 1981.

12. Carey, G. and Oden, J.T., Finite Elements, A Second Course, Volume II, Prentice-Hall, Englewood Cliffs, NJ, 1983.

13. Migliore, H. and McReynolds, E., "Multi-Element Collocation Solution For Convective Dominated Transport," Numerical Methods in Laminar and Turbulent Flow, Proceedings of the Third International Conference, Pineridge Press, Swansea, U.K., pp. 842-852, 1983.

14. Oden, J.T., "Penalty Method and Reduced Integration for the Analysis of Fluids," Penalty-Finite Element Methods in Mechanics, American Society of Mechanical Engineers, AMD-Vol. 51, p. 23, 1982.

15. Gresho, P., Lee, R. and Sani, R., "On the Time-Dependent Solution of the Incompressible Navier-Stokes Equations in Two and Three Dimensions," Lawrence Livermore Lab, Livermore, CA, UCRL-83282, 1979.

16. White, F., Viscous Fluid Flow, McGraw-Hill, New York, NY, 1974.

17. Hindmarsh, A., "ODEPACK, A Systematized Collection of ODE Solvers," Lawrence Livermore Lab, Livermore, CA, UCRL-88007, 1982.

18. Migliore, H.J. and McReynolds, E., "A Collocation Solution To the Two-D Unsteady Navier-Stokes Equations," Forum on Unsteady Flows, Winter Annual Meeting, ASME, 1984.

19. Frind, E., and Pinder, G., "A Collocation Finite Element Method for Potential Problems in Irregular Domains," International Journal for Numerical Methods in Engineering, Vol. 14, pp. 681-701, 1979.

20. Hughes, T.J.R., and Brooks, A., "A Theoretical Framework for Petrov-Galerkin Methods with Discontinuous Weighting Functions: Application to the Streamline-Upwind Procedure," Finite Elements In Fluids, Vol. 4, pp. 47-65, John Wiley, New York, NY, 1982.

21. Migliore, H.J., McReynolds, E.G., and Ray, E.T., "C^0 Collocation Solution For Navier-Stokes Equations With Interactive Boundary Conditions," Submitted to AIAA Journal, 1984.

NOMENCLATURE

A, A_{ij}	x-derivative matrix
B, B_{ij}	y-derivative matrix
a_k	coefficients in general solution
a_α^i	equal to $A_{\alpha i}$, accounts for $\frac{\partial \psi_i}{\partial x}$

b_α^i	equal to $B_{\alpha i}$, accounts for $\frac{\partial\psi_i}{\partial y}$
C^0	zero'th derivative continuity
C^1	first derivative continuity
D	total derivative
e	elements
$\underset{\sim}{i}\ \underset{\sim}{j}$	x,y unit vectors
$\bar{i},\bar{j},k$	subscripts designating variables evaluated at collocation points
$\lvert J\rvert$	determinant of Jacobian matrix
M^{ik}	$W_j\phi_{ij}\phi_{jk}$
$m_{\beta i}M^{ik}$	δ_β^k ; $m = M^{-1}$
$\underset{\sim}{n}$	normal vector to $\partial\Omega$
$\tilde{N},N_i$	shape function
P_k	Legendre polynomials
p	pressure
R	residual
s	integration variable associated with $\partial\Omega$
$\underset{\sim}{T}$	surface traction matrix
$\bar{T}_{ij}$	components of $\underset{\sim}{T}$
t	time
u(x)	one dimensional field variable, such as velocity
$\underset{\sim}{u}$	velocity vector
$\underset{\sim}{u},v$	velocity components in x,y direction
w_k	Lobatto quadrature weights
W_i	master element integration weights
x,y	global, rectangular coordinates
x_i, y_i	Lobatto quadrature points
β	boundary integral
δ_β^k	Kronecker delta
θ	angle of flap rotation, measure clockwise from vertical, in radians
$\dot{\theta}$	angular velocity of flap, in radians/second
λ	large number, penalty parameter
μ	viscosity
υ	kinematic viscosity
ρ	fluid density
ξ,η	element coordinates
ϕ_i	bilinear shape functions
ψ_i	biquadratic shape functions
Ω	spatial domain
$\partial\Omega$	boundary of domain, Ω
∇	Two-D grad operator
$\nabla\cdot\underset{\sim}{T}$	vector surface force

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